YDK 517.9

PROBLEMS OF GEOMETRICALLY NONLINEAR ELASTICITY

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A regularization of the equations of nonlinear elasticity is introduced, and the existence of a solution of the regularized problem is proved for a wide class of data under the "displacement" and "mixed" formulations. The uniqueness is established for small data, and convergence to the solution of non-regularized problem is proved in the case when there exists a solution of non-regularized problem.

1. Initial and regularized problems. Let Ω be a bounded domain in R^n occupied with an elastic body before the deformation. The problem of elasticity consists in finding a vector-function of displacements $u = (u_1, \ldots, u_n)$ such that

$$-\frac{\partial}{\partial x_j}(\sigma_{ij}(u) + \sigma_{qj}(u)\frac{\partial u_i}{\partial x_q}) = f_i \text{ in } \Omega \; i = 1, \ldots, n. \tag{1.1}$$

Here and below the summation over repeated index is implied, f_i are the components of the body force function $f = (f_1, \ldots, f_n)$, $\sigma_{ij}(u)$ are the components of the stress tensor $\sigma(u) = (\sigma_{ij}(u))$

$$\sigma_{ij}(u) = a_{ijkm}\varepsilon_{km}(u), \tag{1.2}$$

where $\varepsilon_{km}(u)$ are the components of the deformation tensor $\varepsilon(u) = (\varepsilon_{km}(u))$

$$\varepsilon_{km}(u) = e_{km}(u) + \frac{1}{2}\frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_m}, \; e_{k,m} = \frac{1}{2}(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}). \tag{1.3}$$

We consider two types of the boundary conditions. The first is the displacement formulation:

$$u\big|_S = 0, \tag{1.4}$$

where S is the boundary of Ω. The second is the mixed formulation. Let S_1 and S_2 be open non-empty sets in S such that $S = \overline{S}_1 \cup \overline{S}_2$, $S_1 \cap S_2 = \emptyset$. For the mixed formulation the boundary conditions are the following

$$u\big|_{S_1} = 0, \; (\sigma_{ij}(u) + \sigma_{qj}(u)\frac{\partial u_i}{\partial x_q})\nu_j\big|_{S_2} = g_i \quad i = 1, \ldots, n, \tag{1.5}$$

where ν_j are the components of the unit outward normal vector $\nu = (\nu_1, \ldots, \nu_n)$ along the boundary S,g_i are the components of the surface force $g = (g_1, \ldots, g_n)$. The displacement formulatiom is the obtained from the mixed when S_2 is an empty set. We suppose that the coefficients of elasticity a_{ijkm} satisfy the conditions

$$\begin{gathered} a_{ijkm} \in L_\infty(\Omega), \; a_{ijkm} = a_{jikm} = a_{ijmk} = a_{mkij}, \\ a_{ijkm}(x)\xi_{ij}\xi_{km} \geq c_0 \sum_{i,j=1}^{n} \xi_{ij}^2 \textit{ almost everywhere} \\ \textit{in } \Omega \; \forall \xi_{ij} = \xi_{ji} \in R, \; c_0 = \text{const} > 0. \end{gathered} \tag{1.6}$$

Define the space V and the operator $N : V \to V^*$, where V^* is the dual of V, as follows

$$V = \left\{ u = (u_1, \ldots, u_n) \in W_4^1(\Omega), \; u\big|_{S_1} = 0 \right\}, \tag{1.7}$$

From the physical viewpoint the nonlinear addends in the deformation tensor $\varepsilon(u) =$
$= (\varepsilon_{km}(u))$ (see (1.3)) are defined by the average angles of deformed elements turning (see [4,5]) and, in general, all mechanics of continuum is based on the averaging. So we define a modified strain tensor $\gamma(v) = (\gamma_{km}(v))$ of the form

$$\gamma_{km}(v) = e_{km}(v) + \frac{1}{2}\frac{\partial(P_\rho v_l)}{\partial x_k}\frac{\partial(P_\rho v_l)}{\partial x_m}. \tag{1.14}$$

Here we consider ρ to be a fixed constant, $\rho \in (0, \alpha)$. Then the strain energy $\Phi_1(v)$ and the components of the stress tensor $\tau(v) = (\tau_{ij}(v))$ are defined by

$$\Phi_1(v) = \frac{1}{2}\int_\Omega a_{ijkm}\gamma_{km}(v)\gamma_{ij}(v)dx, \tag{1.15}$$

$$\tau_{ij}(v) = a_{ijkm}\gamma_{km}(v). \tag{1.16}$$

Define the space V_1 and the functional Ψ_1 on V_1 as follows

$$V_1 = \left\{v \in H^1(\Omega)^n,\ v|_{S_1} = 0\right\}, \tag{1.17}$$

$$\Psi_1(v) = \Phi_1(v) + \beta\int_\Omega a_{ijkm}e_{km}(v)e_{ij}(v)dx - \\ - \int_\Omega f_i v_i dx - \int_{S_2} g_i v_i ds. \tag{1.18}$$

Here β is a small positive constant. Providing $\beta = 0$, the functional Ψ_1 defines the total energy. We define a norm on V_1 by

$$\|v\|_1 = \Big(\int_\Omega a_{ijkm}e_{km}(v)e_{ij}(v)dx\Big)^{1/2}. \tag{1.19}$$

Due to (1.6) and Korn's inequality, this norm is equivalent to the norm of $H^1(\Omega)^n$. Consider the problem:

$$\textit{find } u \textit{ satisfying } u \in V_1 \quad \Psi_1(u) = \min_{v\in V_1}\Psi_1(v). \tag{1.20}$$

We call this function u a solution of the regularized problem for the mixed formulation. Certainly, when S_2 is an empty set, i.e. $S = S_1$, u is a solution of the regularized problem for the displacement formulation. It follows from (1.20) that $\frac{d}{dt}\Psi_1(u+tv)\big|_{t=0} = 0\ \forall v \in V_1$, therefore

$$u \in V_1\ (N_1(u), v) = \int_\Omega f_i v_i dx + \int_{S_2} g_i v_i ds\ \forall v \in V_1, \tag{1.21}$$

where $N_1 : V_1 \to V_1^*$,

$$(N_1(u), v) = \int_\Omega [\tau_{ij}(u)e_{ij}(v) + \tau_{qj}(u)\frac{\partial(P_\rho u_i)}{\partial x_q}\frac{\partial(P_\rho v_i)}{\partial x_j} + \\ + 2\beta a_{ijkm}e_{km}(u)e_{ij}(v)]dx. \tag{1.22}$$

Theorem 1. *Let Ω be a bounded domain in R^n with a Lipschitz continuous boundary S. Let also (1.6), (1.10) hold and $f, g \in V_1^*$, where V_1^* is the dual of V_1. Then for an arbitrary $\rho > 0$, there exists a solution of problem (1.20) that is just a solution of problem (1.21). There exists $r > 0$such that if $||f||_{V_1^*} + ||g||_{V_1^*} \le r$, then the solution of problem (1.20) (respectively (1.21)) is unique.*

Proof. It is easy to see that Ψ_1 is an increasing functional, i.e. $\Psi_1(v) \to \infty$ as $||v||_1 \to \infty$ uniformly with respect to $v \in V_1$. Therefore, if $\{u_n\}$ is a minimizing sequence for Ψ_1 then $\{u_n\}$ is bounded in V_1. So we can choose a subsequence $\{u_m\}$ such that $u_m \to u$ weakly in V_1, and by (1.10), $P_\rho u_m \to P_\rho u$ strongly in $W_4^1(\Omega)^n$. From here we have $\underline{\lim}_{m\to\infty} \Psi_1(u_m) \ge \Psi_1(u)$, $u \in V_1$. Therefore, u is a solution of problem (1.20) and (1.21), respectively.

The functional Ψ_1 is infinitely Fréchet differentiable on V_1, and there exists $\gamma > 0$, such that Ψ_1 is strictly convex in $d_\gamma = \{v \in V_1, ||v||_1 \le \gamma\}$. Then the operator $\mathcal{N}_1$ is strictly monotone in d_γ (see [6]). It can be seen that there exists $r > 0$ such that if $||f||_{V_1^*} + ||g||_{V_1^*} \le r$, then an arbitrary solution of problem (1.20) (respectively (1.21)) belongs to d_r. Therefore, in the case when $||f||_{V_1^*} + ||g||_{V_1^*} \le r$, the solution of problem (1.20) (respectively (1.21)) is unique.

2. Convergence to the solution of non-regularized problem. For the smooth boundary and for small and smooth body forces there exists the unique solution of the initial (non-regularized) problem (1.1), (1.4) (see [2]). We will show that in this case the solutions of regularized problems converge to the solution of problem (1.1), (1.4) as a parameter of regularization tends to zero.

Consider the regularized problem for the displacement formulation. In this case, $S_1 = S$, $V_1 = H_0^1(\Omega)^n$, $P_\rho = P_{1\rho} \circ P_2$. P_2 is an operator of extension on R^n by zero, $P_{1\rho}$ is defined by (1.11). Then we get the following problem:

find u satisfying

$$u \in H_0^1(\Omega)^n \; (\mathcal{N}_1(u), v) = \int_\Omega f_i v_i dx \; \forall v \in H_0^1(\Omega)^n, \tag{2.1}$$

where $(\mathcal{N}_1(u), v)$ is defined by (1.22). By (1.11), (1.16), (1.22) and (2.1) we get the following equations for u

$$-(1+2\beta)\frac{\partial}{\partial x_j}(a_{ijkm}e_{km}(u)) = q_{\rho i} \; i = 1, \ldots, n, \tag{2.2}$$

$$q_{\rho i} = \frac{1}{2}\frac{\partial}{\partial x_j}(a_{ijkm}\frac{\partial(P_\rho v_l)}{\partial x_k}\frac{\partial(P_\rho v_l)}{\partial x_m}) + \frac{\partial}{\partial x_j}(P_\rho^*(\tau_{qj}(u)\frac{\partial(P_\rho u_i)}{\partial x_q})) + f_i. \tag{2.3}$$

Here P_ρ^* is the adjoint of P_ρ operator defined by

$$(P_\rho^*\omega)(x) = \int_{R^n} \Psi_\rho(|x-y|)\omega(y)dy, \tag{2.4}$$

where $\omega \in L_2(\Omega)$ and ω is extended by zero outside of Ω, and equations (2.2) are considered in the sense of distributions.

We suppose that the boundary S is of the class C^2, $a_{ijkm} \in C^1(\overline{\Omega})$ in addition to (1.6), and $f \in L_p(\Omega)^n$, $p > n$. Then $q_{\rho i} \in L_p(\Omega)$ and from [7,8] it follows that $u \in W_p^2(\Omega)^n$.

Now let $\{P_\rho\}, \rho \in (0, a]$ be a family of regularizing operators such that

$$\lim_{\rho\to 0} ||P_\rho\omega - \omega||_{W_p^2(\Omega)} = 0 \; \forall\omega \in W_p^2(\Omega). \tag{2.5}$$

The operator P_ρ has the form $P_\rho = P_{1\rho} \circ P$, where P is an operator of extension on R^n, $P \in \mathcal{L}(W_p^2(\Omega), W_p^2(R^n))$, $P_{1\rho}$ is defined by (1.11). We define the space V_2 as follows

$$V_2 = W_p^2(\Omega)^n \cap \overset{\circ}{H}{}^1(\Omega)^n, \; p > n. \tag{2.6}$$

V_2 is a Banach space with the norm of $W_p^2(\Omega)^n$, and we denote this norm by $||\cdot||_2$. Let us consider the problem

$$u_\rho \in V_2, \quad A_\rho(u_\rho) = f, \tag{2.7}$$

$$A_\rho(u_\rho) = \{A_\rho(u_\rho)_i\}_{i=1}^n,$$
$$A_\rho(u_\rho)_i = -(1 + 2\beta(\rho))\frac{\partial}{\partial x_j}(a_{ijkm}e_{km}(u_\rho)) -$$
$$-\frac{1}{2}\frac{\partial}{\partial x_j}\left(a_{ijkm}\frac{\partial(P_\rho v_l)}{\partial x_k}\frac{\partial(P_\rho v_l)}{\partial x_m}\right) - \frac{\partial}{\partial x_j}\left(P_\rho^*\left(\tau_{qj}(u_\rho)\frac{\partial(P_\rho u_i)}{\partial x_q}\right)\right). \tag{2.8}$$

Here $\tau_{qj}(u_\rho)$ is defined by (1.14), (1.16), where P_ρ is such that (2.5) holds, P_ρ^* is defined by (2.4), where ω is extended by zero outside of Ω, and β is considered as a following function on ρ

$$\begin{cases} \beta \text{ is a continuous function decreasing on} [0, \alpha], \\ \beta(\rho) > 0 \; \forall \beta \in (o, \alpha], \; \beta(0) = 0. \end{cases} \tag{2.9}$$

Theorem 2. *Let Ω be a bounded domain in R^n with a boundary S of the class C^2, $a_{ijkm} \in C^1(\overline{\Omega})$ and (1.6), (2.5), (2.9) hold. Let also $f \in L_p(\Omega)^n$, $p > n$. Then there exist positive constants r, ρ_0 such that for $f \in d_r = \{f \in L_p(\Omega)^n, \; ||f||_{L_p(\Omega)^n} \leq r\}$, $\rho \in (0, \rho_0]$, there exists a unique solution of problem (2.7). The function $\rho \rightarrow u_\rho$ is a continuous mapping from $(0, \rho]$ into V_2, and $u_\rho \rightarrow u$ in $W_p^2(\Omega)^n$ as $\rho \rightarrow 0$, where u is a solution of non-regularized problem (1.1), (1.4).*

Here we sketch a proof of Theorem 2. The function A_ρ a continuously Fréchet differentiable mapping from V_2 into $L_p(\Omega)^n$, and its derivative $A'_\rho(v)$ at a point v has the form

$$A'_\rho(v) = J_\rho + U_\rho(v). \tag{2.10}$$

Here J_ρ is the operator of linear elasticity

$$J_\rho\omega = \left\{-(1 + 2\beta(\rho))\frac{\partial}{\partial x_j}(a_{ijkm}e_{km}(\omega))\right\}_{i=1}^n, \tag{2.11}$$

and $||U_\rho(v)||_{\mathcal{L}(V_2, L_p(\Omega)^n)} \rightarrow 0$ as $||v||_2 \rightarrow 0$ uniformly with respect to $\rho \in (0, \rho_0]$. It follows from [7,8] that J_ρ is an isomorphism from V_2 onto $L_p(\Omega)^n$.

We define the mapping $G_\rho : L_p(\Omega)^n \times V_2$ of the form

$$G_\rho(f, v) = v - J_\rho^{-1}(A_\rho(v) - f) \tag{2.12}$$

and prove that there exist positive constants r, γ, ρ_0 such that $\forall \rho \in (0, \rho_0]$ and $\forall f \in d_r$ the mapping $G_\rho(f, \cdot) : v \rightarrow G_\rho(f, v)$ is a contraction in $\tilde{d}_\gamma = \{v \in V_2, \; ||v||_2 \leq \gamma\}$, i.e

$$||G_\rho(f, v) - G_\rho(f, w)||_2 \leq c_1||v - \omega||_2$$
$$\forall v, w \in \tilde{d}_\gamma \; \forall \rho \in [0, \rho_0], \; c_1 < 1. \tag{2.13}$$

For $\rho = 0$ we have $P_0 = I$, where I is the identical operator in $W_p^2(\Omega)$. The function $G_\rho(f,\cdot)$ maps $\tilde{d}_\gamma$ into $\tilde{d}_\gamma$. Therefore for an arbitrary $\rho \in [0,\rho_0]$, there exists a unique $u_\rho \in \tilde{d}_\gamma$ such that $u_\rho = G_\rho(f,u_\rho)$, $u_0 = u$ being a solution of problem (1.1), (1.4). We have

$$||u_\rho - u_0||_2 = ||G_\rho(f,u_\rho) - G_0(f,u_0)||_2 \leq$$
$$\leq ||G_\rho(f,u_\rho) - G_\rho(f,u_0)||_2 + ||G_\rho(f,u_0) - G_0(f,u_0)||_2. \tag{2.14}$$

By (2.13), (2.14) we get

$$||u_\rho - u_0||_2 \leq (1-c_1)^{-1}||G_\rho(f,u_0) - G_0(f,u_0)||_2.$$

The right hand side of this inequality tends to zero as $\rho \to 0$. Therefore $u_\rho \to u_0$ in $W_p^2(\Omega)^n$ as $\rho \to 0$.

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